TELEPHONE No.
20251.

THE UNIVERSITY, LEEDS.

11th December, 1934.

Dr. Armin Weiner,
Nova 36,
BRUENN.

Dear Dr. Weiner,

Thank you for your kind letter of the 26th November. You will, I think, already have heard officially from the Zionist Office that it is, unfortunately, not possible for me to visit Bruenn, much as I should have liked to do so.

I enclose, in accordance with your request, the manuscript of an article of mine on "The Line of Action of the Resultant Pressure in Discontinuous Fluid Motion" published in 1922.

Yours sincerely,

S. BRODETSKY.

Enc.

ISO

Proceedings AHW Copy 18.10.22 Becker

(87)

THE LINE OF ACTION OF THE RESULTANT PRESSURE IN DISCONTINUOUS FLUID MOTION.

S. Brodetsky, M.A., Ph.D.

Rec'd June 7 Communicated by Sir George Greenhill, F.R.S.

1. The problem of any barrier in a fluid stream is best attacked by the method due to Levi-Civita (Rend. del Circ. di Palermo, XXIII, Nov. 1906), of which useful accounts, with extensions, are given by Cisotti (Idromeccanica Piana, II, Milano, 1922) and Brillouin (Ann. de Chim. et Phys., XXIII, 1911, 145-230). The resultant pressure for any barrier has been given in terms of the constants defining the barrier: but the calculations required to find the line of action of this pressure have not been carried out. It is our object to supply this deficiency here.

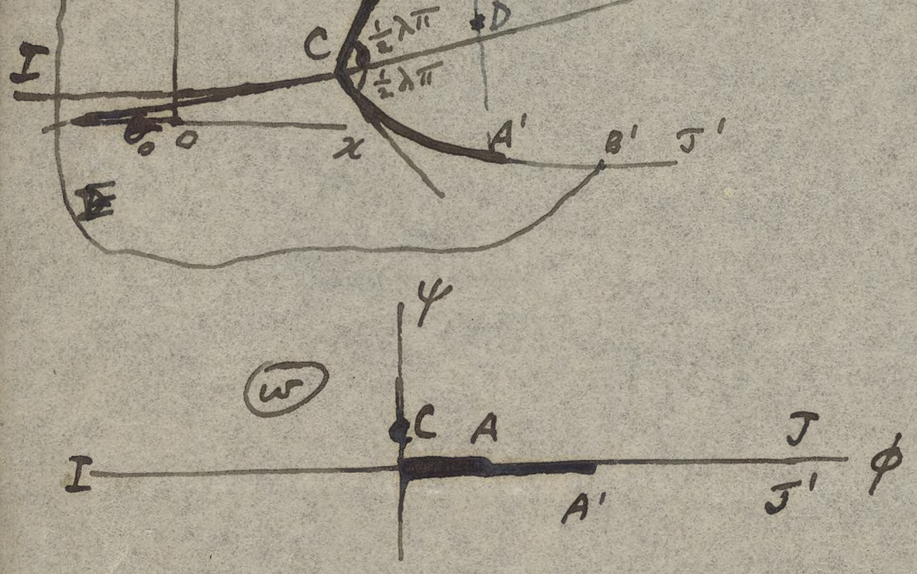
The motion is two-dimensional. Let the complex variable $z(\equiv x+iy)$ define position in any plane perpendicular to the generators of the barrier, the x axis being parallel to the direction of the stream at infinity. We define

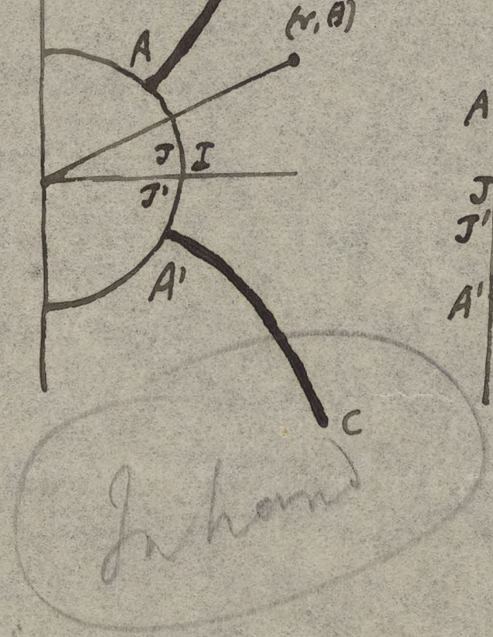
$$u = \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}, \qquad v = \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}$$

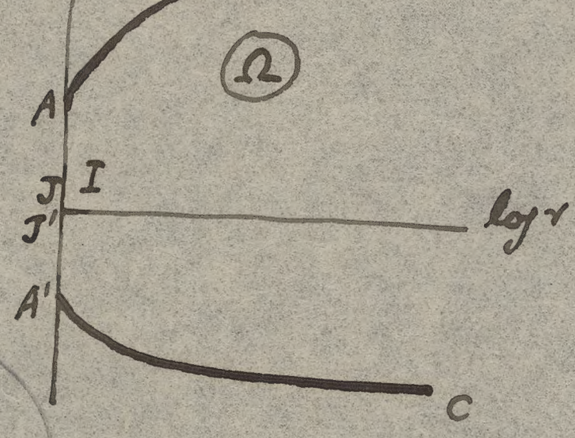
where u, v are the velocity components, and ϕ, ψ are the velocity potential and stream function respectively. Let $w \equiv \phi + i\psi$ and define ζ, Ω, r, θ so that

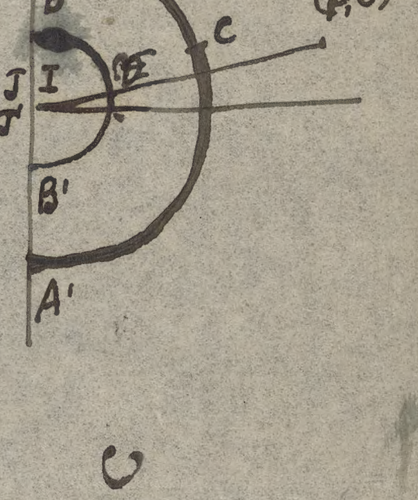
$$\zeta \equiv re^{i\theta} = \frac{dz}{dw}\ ; \quad \Omega = \log\zeta \equiv \log r + i\theta \qquad \ldots \qquad (1)$$

The figure shows the z, w, ζ, Ω planes for such a problem, with the boundary conditions. C is the point of bifurcation of the stream line IC, CA, CA' are the stream-lines in contact with the barrier, $AJ, A'J'$ are the free stream-lines. We take the standard dimensions to be unit velocity at infinity, unit value of ~~the distance~~ $\frac{1}{2}(\sqrt{CA}+\sqrt{CA'})$ as measured in the w plane, and unit density. For other values of these quantities we multiply lengths in the z plane by the square of $\frac{1}{2}(\sqrt{CA}+\sqrt{CA'})$ in the w plane, and we multiply the resultant pressure per unit length of the barrier by the square of the velocity at infinity, by the density, and by the square of $\frac{1}{2}(\sqrt{CA}+\sqrt{CA'})$ in the w plane.



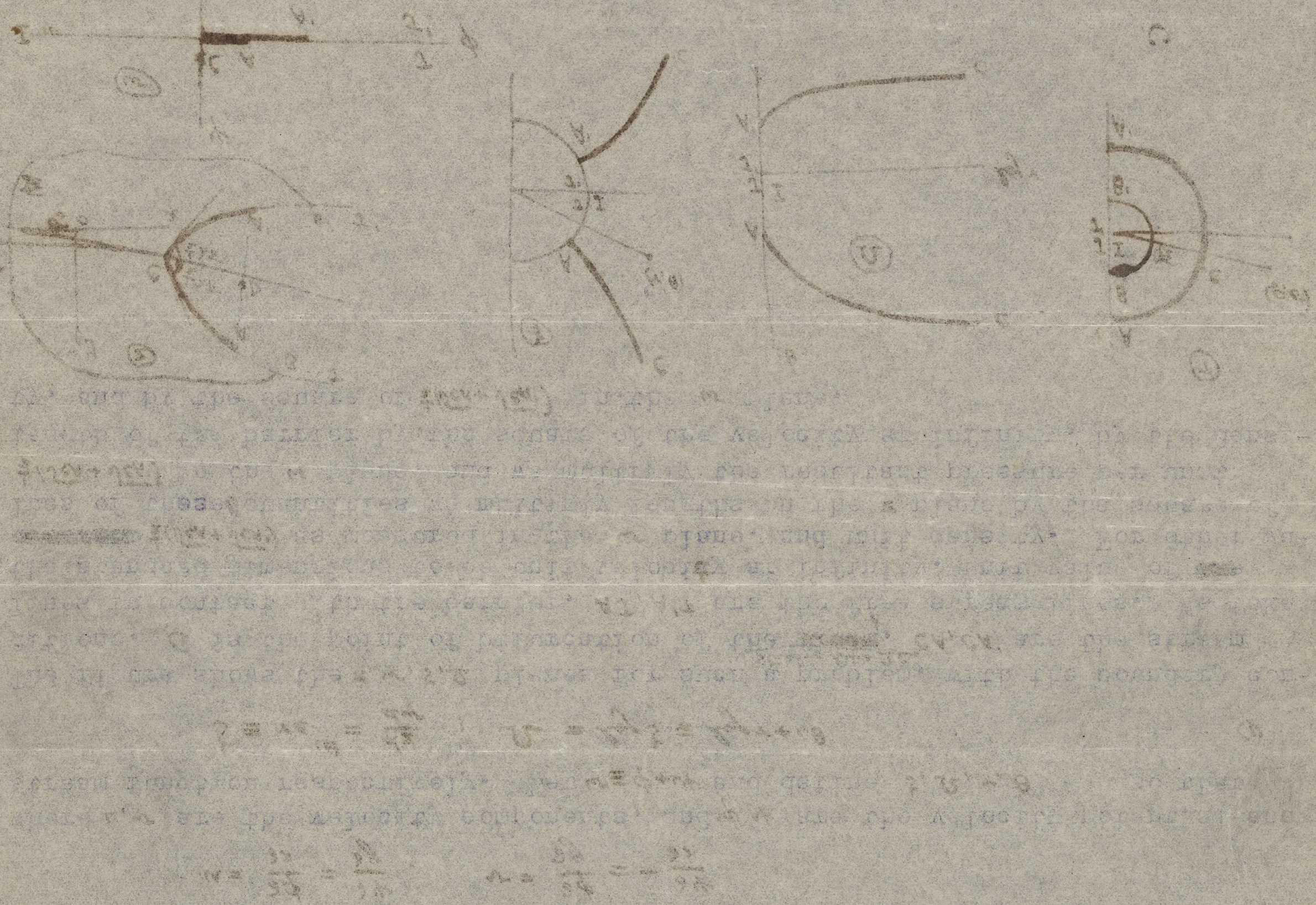






We introduce the transformation in which the variable $T \equiv \rho e^{i\sigma}$ is given by

$$\sqrt{w} = \frac{1}{2c}\left(T - \frac{1}{T}\right) - \sin\sigma_0, \qquad \ldots \qquad (2)$$



DS

where the point C is given by $\tau = e^{i\sigma_0}$. It is immediately verified that the barrier ACA' is the semicircle $\rho = 1, -\frac{\pi}{2} \le \sigma \le +\frac{\pi}{2}$ in the τ plane, while the free stream lines $AJ, A'J'$ become the two halves of the diameter along the imaginary axis, namely $\tau = i\rho, -1 \le \rho \le +1$.

If the barrier is plane we have

$$\Omega = \log\left(\frac{1+e^{i\sigma_0\tau}}{1-e^{-i\sigma_0\tau}}\right), \quad \zeta = \frac{1+e^{i\sigma_0\tau}}{1-e^{-i\sigma_0\tau}} \qquad \text{(3)}$$

If it consists of two planes meeting at an angle $\lambda\pi$ at C (measured away from the streaming fluid), then

$$\Omega = \lambda \log\left(\frac{1+e^{i\sigma_0\tau}}{1-e^{-i\sigma_0\tau}}\right), \quad \zeta = \left(\frac{1+e^{i\sigma_0\tau}}{1-e^{-i\sigma_0\tau}}\right)^{\lambda} \qquad \text{(4)}$$

In the general problem we write

$$\Omega = \lambda \log\left(\frac{1+e^{i\sigma_0\tau}}{1-e^{-i\sigma_0\tau}}\right) + A_1\tau + \tfrac{1}{2}iA_2\tau^2 + \tfrac{1}{3}A_3\tau^3 + \tfrac{1}{4}iA_4\tau^4 + \cdots \qquad \text{(5)}$$

$$\equiv a_1\tau + \tfrac{1}{2}ia_2\tau^2 + \tfrac{1}{3}a_3\tau^3 + \tfrac{1}{4}ia_4\tau^4 + \cdots \qquad \text{(6)}$$

where the A's, a's are all real, and the expansion (5) is convergent for $|\tau| \le 1$. The angle between the tangents at C in the z plane is again $\lambda\pi$, while the bisector of this angle makes with the x axis the angle

$$\lambda\sigma_0 + A_1\sin\sigma_0 + \tfrac{1}{2}A_2\cos 2\sigma_0 + \tfrac{1}{3}A_3\sin 3\sigma_0 + \tfrac{1}{4}A_4\cos 4\sigma_0 + \cdots .$$

If then we have any set of constants we can find the barrier and its position.

2. In our notation the components of pressure on unit length of the barrier, R_x, R_y, and the moment M about the origin in the z plane, are given by

$$\left.\begin{aligned} R_y + iR_x &= \tfrac{1}{2}\int_{B'EB} \frac{dz}{\zeta^2} - \tfrac{1}{2}\int_{B'EB}(dx - i\,dy), \\ M &= \text{real part of } \left\{\tfrac{1}{2}\int_{B'EB}\frac{z\,dz}{\zeta^2} - \tfrac{1}{4}\left[\int(dx+i\,dy)\int(dx-i\,dy)\right]_{B'}^{B}\right\} \end{aligned}\right\} \cdots \qquad (7)$$

the integration referring to any contour like $B'EB$, where B' is on $A'J'$, B is on AJ, and the contour lies wholly within the moving fluid. If we express the integrals in terms of τ we can take for our contour any curve like $B'EB$ (see τ plane in the figure). It is well known that

$$R_x = \frac{\pi}{4}a_1^2, \quad R_y = \frac{\pi}{4}(a_2 + 4a_1\sin\sigma_0) \qquad \text{(8)}$$

(see Cisotti, p. 175, Brillouin, p. 195).

As regards the moment Cisotti does not go beyond giving a formula like (7), while Brillouin makes a few general remarks: neither works out the explicit value of M, and neither defines the meaning of the term "origin" in this connection.

Since the solution (5, 6) gives us really the value of $dz/d\tau$, and we have to integrate in order to get z in terms of τ, it is clear that no origin has yet been defined in the z plane. We shall define the origin as follows. Let $dz/d\tau$ be expanded in terms of τ: we shall soon see that the expansion contains terms in $\tau^{-3}, \tau^{-2}, \tau^{-1}, \tau^0, \tau^1, \tau^2 \ldots$. Integrate the series without introducing any arbitrary constant, so that we get terms in $\tau^{-2}, \tau^{-1}, \log\tau, \tau', \tau^2, \tau^3 \ldots$. We take the origin to correspond to the value of τ that makes the expansion thus obtained vanish.

3. If we use the fact that $dz/dw = \zeta$, and let the symbol $(\ )_-$ denote that in

the expression enclosed by the brackets, $-\iota$ is to be substituted for $+\iota$, we can write M in the form

$$M = \underline{\text{real part of}} \left\{ \frac{1}{2} \int_{B'EB} \frac{1}{\zeta} \frac{d\omega}{d\tau} \left(\int \zeta \frac{d\omega}{d\tau} d\tau \right) d\tau - \frac{1}{4} \left[\left(\int \zeta \frac{d\omega}{d\tau} d\tau \right) \left(\int \zeta \frac{d\omega}{d\tau} d\tau \right)_- \right]_{B'}^{B} \right\} \quad \cdots \quad (9)$$

But

$$\zeta = \exp\left(a_1 \tau + \tfrac{1}{2} \iota a_2 \tau^2 + \tfrac{1}{3} a_3 \tau^3 + \cdots \right),$$

so that $1/\zeta$ is obtained from ζ by changing the signs of all the a's. Also

$$\frac{d\omega}{d\tau} = \frac{1+\tau^2}{2\tau^3}\left(1 + 2\iota \sin\sigma_0 \cdot \tau - \tau^2\right) = \frac{1+\tau^2}{2\tau^3}\left(1 + e^{\iota\sigma_0}\tau\right)\left(1 - e^{-\iota\sigma_0}\tau\right) \quad \cdots \quad (10)$$

Hence it follows that $\zeta\, d\omega/d\tau$ begins with a term τ^{-3}. Further, if we use the notation

$$\zeta \frac{d\omega}{d\tau} = -\frac{2(\alpha_{-2} + \iota\beta_{-2})}{\tau^3} - \frac{(\alpha_{-1} + \iota\beta_{-1})}{\tau^2} + \frac{(\alpha_0 + \iota\beta_0)}{\tau} + (\alpha_1 + \iota\beta_1) + 2(\alpha_2 + \iota\beta_2)\tau + \cdots \quad \cdots \quad (11)$$

we at once see that $\zeta^{-1} d\omega/d\tau$ is got by changing the signs of β, α alternately, i.e.

$$\frac{1}{\zeta}\frac{d\omega}{d\tau} = -\frac{2(\alpha_{-2} - \iota\beta_{-2})}{\tau^3} + \frac{(\alpha_{-1} - \iota\beta_{-1})}{\tau^2} + \frac{(\alpha_0 - \iota\beta_0)}{\tau} - (\alpha_1 - \iota\beta_1) + 2(\alpha_2 - \iota\beta_2)\tau - \cdots \quad \cdots \quad (12)$$

Again, by the definition of the origin, (11) gives

$$\int \zeta \frac{d\omega}{d\tau} d\tau = \frac{(\alpha_{-2} + \iota\beta_{-2})}{\tau^2} + \frac{(\alpha_{-1} + \iota\beta_{-1})}{\tau} + (\alpha_0 + \iota\beta_0)\log\tau + (\alpha_1 + \iota\beta_1)\tau + (\alpha_2 + \iota\beta_2)\tau^2 + \cdots \quad \cdots \quad (13)$$

$$= \frac{(\alpha_{-2} + \iota\beta_{-2})}{\rho^2} e^{-2\iota\sigma} + \frac{(\alpha_{-1} + \iota\beta_{-1})}{\rho} e^{-\iota\sigma} + (\alpha_0 + \iota\beta_0)(\log\rho + \iota\sigma) + (\alpha_1 + \iota\beta_1)\rho e^{\iota\sigma} + (\alpha_2 + \iota\beta_2)\rho^2 e^{2\iota\sigma} + \cdots \quad (14)$$

Hence we deduce that

$$\left(\int \zeta \frac{d\omega}{d\tau} d\tau \right)_- = \frac{(\alpha_{-2} - \iota\beta_{-2})}{\rho^2} e^{2\iota\sigma} + \frac{(\alpha_{-1} - \iota\beta_{-1})}{\rho} e^{\iota\sigma} + (\alpha_0 - \iota\beta_0)(\log\rho - \iota\sigma) + (\alpha_1 - \iota\beta_1)\rho e^{-\iota\sigma} + (\alpha_2 - \iota\beta_2)\rho^2 e^{-2\iota\sigma} + \cdots \quad (15)$$

If we form the product of (14), (15) we get terms in $\rho^{-4}, \rho^{-3}, \rho^{-2}, \rho^{-1}, \log\rho, \rho^0, \rho^1, \rho^2, \ldots$.

Now let us take the contour $B'EB$ in the τ plane to be a semicircle whose radius ρ is very small, so that $\rho \to 0$. We can then omit all terms with positive powers of ρ. Also the negative powers of ρ and the $\log\rho$ term must necessarily cut out with the corresponding terms in the other part of M. Thus in the product $\left(\int \zeta \frac{d\omega}{d\tau} d\tau \right)\left(\int \zeta \frac{d\omega}{d\tau} d\tau \right)_-$ we need only take the terms independent of ρ, i.e.

$$2(\alpha_2\alpha_{-2} + \beta_2\beta_{-2})\cos 4\sigma + 2(\alpha_2\beta_{-2} - \alpha_{-2}\beta_2)\sin 4\sigma + 2(\alpha_1\alpha_{-1} + \beta_1\beta_{-1})\cos 2\sigma + 2(\alpha_1\beta_{-1} - \alpha_{-1}\beta_1)\sin 2\sigma + (\alpha_0^2 + \beta_0^2)\sigma^2 ;$$

and if we put successively $\sigma_0 = \pi/2, \sigma_0 = -\pi/2$ and subtract, we get zero.

To get the other part of M we have to multiply (12) into (13), and then integrate. We get terms in $\tau^{-5}, \tau^{-4}, \tau^{-3}, \tau^{-2}, \tau^{-1}, \tau^0, \tau^1, \tau^2, \ldots$; $\tau^{-3}\log\tau, \tau^{-2}\log\tau, \tau^{-1}\log\tau, \log\tau, \tau\log\tau, \tau^2\log\tau, \ldots$. When we integrate we get terms in $\tau^{-4}, \tau^{-3}, \tau^{-2}, \tau^{-1}, \log\tau, \tau^1, \tau^2, \ldots$; $\tau^{-2}(\log\tau + \tfrac{1}{2}), \tau^{-1}(\log\tau + 1), (\log\tau)^2, \tau(\log\tau - 1), \ldots$. If now we make $\tau = \rho e^{\iota\sigma}$ and let ρ be constant and small, so that $\rho \to 0$, the positive powers of ρ can be omitted, and the negative powers of ρ as well as the $\log\rho$ parts of the terms $\log\tau, (\log\tau)^2$ must be cancelled by corresponding terms in the part of M already considered. This can in fact be verified straightforwardly, but it is not worth while reproducing the rather complicated algebra here. The terms we have to consider are thus seen to be

$$\left[4c(\alpha_2\beta_{-2}-\alpha_{-2}\beta_2)-2c(\alpha_1\beta_{-1}-\alpha_{-1}\beta_1)\right]\log\tau+(\alpha_0^2+\beta_0^2)(\log\tau)^2,$$

and we find that

$$M=\left[\left\{(\alpha_1\beta_{-1}-\alpha_{-1}\beta_1)-2(\alpha_2\beta_{-2}-\alpha_{-2}\beta_2)\right\}\sigma-(\alpha_0^2+\beta_0^2)\sigma^2\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$=\pi\left[(\alpha_1\beta_1-\alpha_1\beta_1)-2(\alpha_2\beta_{-2}-\alpha_{-2}\beta_2)\right]$$

The form of M shows that we need find only the first five terms in (11), so that we have to expand ζ only as far as τ^4. If we do this and substitute in M we get

$$M=\frac{\pi}{32}\left\{3a_1^2a_2-a_4+4\sin\sigma_0(a_1^3+2a_1-2a_3)+16a_2\sin^2\sigma_0\right\} \quad \ldots \quad (16)$$

where, in terms of the A'_0, we have

$$a_1=A_1+2\lambda\cos\sigma_0,\quad a_2=A_2-2\lambda\sin 2\sigma_0,\quad a_3=A_3+2\lambda\cos 3\sigma_0,\quad a_4=A_4-2\lambda\sin 4\sigma_0,\ \ldots$$

4. If the barrier is plane then

$$\lambda=1,\quad A_1=A_2=\ldots=0;$$

so that by (8), (16) we have

$$R_x=\pi\cos^2\sigma_0,\quad R_y=\pi\sin\sigma_0\cos\sigma_0,\quad M=-\frac{\pi}{4}\sin\sigma_0\cos\sigma_0.$$

The resultant pressure is therefore $\pi\cos\sigma_0$ perpendicular to the plane, and its distance from the origin is

$$\tfrac{1}{4}\sin\sigma_0$$

in such a sense that the moment is negative, i.e. above the origin in our figure when σ_0 is positive. We can find the origin quite easily. By (3), (10)

$$\frac{dz}{d\tau}=\frac{1+\tau^2}{2\tau^3}\left(1+e^{i\sigma_0}\tau\right)^2,$$

and the origin is given by that value of τ for which

$$\tfrac{1}{4}\left(e^{i\sigma_0}\tau^2-\frac{e^{-i\sigma_0}}{\tau^2}\right)+\left(\tau-\tfrac{1}{\tau}\right)+\cos\sigma_0\cdot\log\tau=0$$

so that $\rho=1$ and σ is given by the equation

$$\tfrac{1}{2}\sin(2\sigma+\sigma_0)+2\sin\sigma+\sigma\cos\sigma_0=0$$

The origin is thus on the barrier when it is plane.

It is not convenient to solve the equation for σ, and in fact this is not necessary. The general value of z in terms of τ is

$$ze^{-i\sigma_0}=\tfrac{1}{4}\left(e^{i\sigma_0}\tau^2-\frac{e^{-i\sigma_0}}{\tau^2}\right)+\left(\tau-\tfrac{1}{\tau}\right)+\cos\sigma_0\cdot\log\tau.$$

Hence for the points A, A' , at which $\tau = +1, \tau = -1$ respectively, we have

$$z_A = ie^{i\sigma_0}(2 - \tfrac{1}{2}\sin\sigma_0 + \tfrac{\pi}{2}\cos\sigma_0) \ , \ z_{A'} = -ie^{i\sigma_0}(2 + \tfrac{1}{2}\sin\sigma_0 + \tfrac{\pi}{2}\cos\sigma_0).$$

It follows that the width of the barrier is

$$4 + \pi\cos\sigma_0 ,$$

while the distance of the origin from the mid-line of the barrier is

$$\tfrac{1}{2}\sin\sigma_0$$

towards A. Hence the resultant pressure meets the barrier at distance

$$\tfrac{3}{4}\sin\sigma_0$$

from the mid-line measured towards A . If then the width of the plane barrier is l, the distance of the centre of pressure from the mid-line is

$$\tfrac{3}{4}l\,\frac{\sin\sigma_0}{4+\pi\cos\sigma_0}$$

the well known result.

It is of interest to note that the ratio of the distances of the origin and of the centre of pressure from the mid-line is $2:3$ for all values of σ_0. When $\sigma_0 = 0$, the plane barrier being normal to the stream, the two points coincide on the mid-line.

5. In more general cases the expansion of $\tfrac{1}{2}dw/d\tau$ in terms of τ is often impracticable, and we proceed as follows. Using the notation of (11) we see that

$$\int\left\{\tfrac{1}{2}\frac{dw}{d\tau} + \frac{2(\alpha_{-2}+i\beta_{-2})}{\tau^3} + \frac{(\alpha_{-1}+i\beta_{-1})}{\tau^2} - \frac{(\alpha_0+i\beta_0)}{\tau}\right\}d\tau$$

vanishes when $\tau = 0$. Hence its value for any point τ can be written

$$\int_0^\tau\left\{\tfrac{1}{2}\frac{dw}{d\tau} + \frac{2(\alpha_{-2}+i\beta_{-2})}{\tau^3} + \frac{(\alpha_{-1}+i\beta_{-1})}{\tau^2} - \frac{(\alpha_0+i\beta_0)}{\tau}\right\}d\tau$$

Thus the value of z in terms of τ without any arbitrary constants is

$$z = \frac{(\alpha_{-2}+i\beta_{-2})}{\tau^2} + \frac{(\alpha_{-1}+i\beta_{-1})}{\tau} + (\alpha_0+i\beta_0)\log\tau + \int_0^\tau\left\{\tfrac{1}{2}\frac{dw}{d\tau} + \frac{2(\alpha_{-2}+i\beta_{-2})}{\tau^3} + \frac{(\alpha_{-1}+i\beta_{-1})}{\tau^2} - \frac{(\alpha_0+i\beta_0)}{\tau}\right\}d\tau \quad \ldots\ (17)$$

Now the integrand in the integral in (17) has no singularities inside the circle $|\tau| = 1$, and on the circumference of this circle the only possible singularity is at C, i.e. $\tau = e^{i\sigma_0}$, when $\lambda > 1$. Hence if τ is any point inside or on the circumference of this circle, except C , we can choose for the path of integration in (17) any contour lying within the semicircle defined in the τ plane, and joining the origin $\tau = 0$ to the point τ . Let us choose the straight line lying between the origin $\tau = 0$ and the point τ . If τ is a point on the

circumference of $|\tau|=1$ so that $\rho=1$, $\tau=e^{i\sigma}$, we get

$$z=(\alpha_{-2}+i\beta_{-2})e^{-2i\sigma}+(\alpha_{-1}+i\beta_{-1})e^{-i\sigma}+i(\alpha_0+i\beta_0)\sigma+\int_0^1\left\{\left(\zeta\frac{dw}{d\tau}\right)_{\tau=\rho e^{i\sigma}}+\frac{2(\alpha_{-2}+i\beta_{-2})}{\rho^3}e^{-3i\sigma}+\frac{(\alpha_{-1}+i\beta_{-1})}{\rho^2}e^{-2i\sigma}-\frac{(\alpha_0+i\beta_0)}{\rho}e^{-i\sigma}\right\}e^{i\sigma}\cdot d\rho,$$

where z is now a point on the barrier ACA' in the z plane.

Let $\sigma=\pm\pi/2$, corresponding to the points A, A' respectively, and use the value of $dw/d\tau$ given by (10) : we find

$$z_A=-(\alpha_{-2}+i\beta_{-2})-i(\alpha_{-1}+i\beta_{-1})+\tfrac{1}{2}\pi(\alpha_0+i\beta_0)+\int_0^1\left\{-\frac{1-\rho^2}{2\rho^3}(1-2\rho\sin\sigma_0+\rho^2)\zeta(i\rho)-\frac{2(\alpha_{-2}+i\beta_{-2})}{\rho^3}-\frac{i(\alpha_{-1}+i\beta_{-1})}{\rho^2}-\frac{(\alpha_0+i\beta_0)}{\rho}\right\}d\rho,$$

$$z_{A'}=-(\alpha_{-2}+i\beta_{-2})+i(\alpha_{-1}+i\beta_{-1})-\tfrac{1}{2}\pi(\alpha_0+i\beta_0)+\int_0^1\left\{-\frac{1-\rho^2}{2\rho^3}(1+2\rho\sin\sigma_0+\rho^2)\zeta(-i\rho)-\frac{2(\alpha_{-2}+i\beta_{-2})}{\rho^3}+\frac{i(\alpha_{-1}+i\beta_{-1})}{\rho^2}-\frac{(\alpha_0+i\beta_0)}{\rho}\right\}d\rho$$

where $\zeta(i\rho), \zeta(-i\rho)$ are the values of ζ for $\tau=\pm i\rho$ respectively. The straight line joining AA' in the z plane can be appropriately called the chord. If D is the mid-point of AA' , defining the mid-line of the barrier, we have at once

$$z_D=-(\alpha_{-2}+i\beta_{-2})+\int_0^1\left\{-\frac{1-\rho^2}{4\rho^3}\left[(1-2\rho\sin\sigma_0+\rho^2)\zeta(i\rho)+(1+2\rho\sin\sigma_0+\rho^2)\zeta(-i\rho)\right]-\frac{2(\alpha_{-2}+i\beta_{-2})}{\rho^3}-\frac{(\alpha_0+i\beta_0)}{\rho}\right\}d\rho$$

A little algebra shows that

$$z_D=\frac{1}{4}-\int_0^1\left\{\frac{1-\rho^2}{4\rho^3}\left[(1-2\rho\sin\sigma_0+\rho^2)\zeta(i\rho)+(1+2\rho\sin\sigma_0+\rho^2)\zeta(-i\rho)\right]-\frac{1}{2\rho^3}+\frac{a_1^2+i(a_2+4a_1\sin\sigma_0)}{4\rho}\right\}d\rho, \quad \text{----} \quad (18)$$

and that the value of the integrand in the integral of (18) is zero when $\rho=0$.

If the function $\zeta(\tau)$ can conveniently be divided into an even function of τ and an odd function of τ ; let these be $Z_0(\tau)$, $Z_1(\tau)$ respectively. Then

$$Z_0(-i\rho)=Z_0(i\rho) \quad , \quad Z_1(-i\rho)=-Z_1(i\rho),$$

and equation (18) takes the form

$$z_D=\frac{1}{4}-\int_0^1\left\{\frac{1-\rho^4}{2\rho^3}Z_0(i\rho)-\frac{1-\rho^2}{\rho^2}\sin\sigma_0\cdot Z_1(i\rho)-\frac{1}{2\rho^3}+\frac{a_1^2+i(a_2+4a_1\sin\sigma_0)}{4\rho}\right\}d\rho. \quad \text{----} \quad (19)$$

The coordinates x_D, y_D of the point D are given by (18) or (19) , whichever is more convenient. Thus for the plane barrier we use (3) and we get

$$z_D=\frac{1}{4}(1-e^{2i\sigma_0})=\frac{1}{2}\sin\sigma_0(\sin\sigma_0-i\cos\sigma_0),$$

agreeing with the results already xxxxxxx obtained.

In more complicated problems we can find x_D, y_D by means of numerical or graphical integration, remembering that the integrand in the integral of (18) or (19)

ISO

is zero when $\rho = 0$.

6. Again, by the equations of §5 we have

$$\tfrac{1}{2}(z_A - z_{A'}) = -(\alpha_1 + i\beta_1) + \tfrac{1}{2}\pi(\alpha_0 + i\beta_0) + \int_0^1 \left\{ -\frac{1-\rho^2}{4\rho^3}\left[(1 - 2\rho\sin\sigma_0 + \rho^2)S(\rho) - (1 + 2\rho\sin\sigma_0 + \rho^2)S(-\rho)\right] - \frac{i(\alpha_1 + i\beta_1)}{\rho^2}\right\} d\rho .$$

Hence we find

$$\tfrac{1}{2}(z_A - z_{A'}) = z_A - z_0 = z_0 - z_{A'}$$

$$= -(\sin\sigma_0 - \tfrac{i}{2}a_1) - \frac{\pi}{8}(a_2 + 4a_1\sin\sigma_0 - ia_1^2) - \int_0^1 \left\{ \frac{1-\rho^2}{4\rho^3}\left[(1 - 2\rho\sin\sigma_0 + \rho^2)S(\rho) - (1 + 2\rho\sin\sigma_0 + \rho^2)S(-\rho)\right] + \frac{\sin\sigma_0 - \frac{i}{2}a_1}{\rho^2}\right\} d\rho, \quad (20)$$

or

$$= -(\sin\sigma_0 - \tfrac{i}{2}a_1) - \frac{\pi}{8}(a_2 + 4a_1\sin\sigma_0 - ia_1^2) - \int_0^1 \left\{ \frac{1-\rho^4}{2\rho^3} Z_1(\rho) - \frac{1-\rho^2}{\rho^2}\sin\sigma_0 \cdot Z_0(\rho) + \frac{\sin\sigma_0 - \frac{i}{2}a_1}{\rho^2}\right\} d\rho, \quad \ldots (21)$$

where Z_0, Z_1 have the same meanings as in § 5, and the value of the integrand in (20) or (21) when $\rho = 0$ is

$$\tfrac{1}{12}(6a_1^2\sin\sigma_0 + 12\sin\sigma_0 + 3a_1a_2 + 6ia_2\sin\sigma_0 - 2ia_3 - ia_1^3) \quad \ldots \quad (22)$$

We can thus find the positions of A, A' relatively to D in the z plane, and also the origin. The barrier itself can be drawn by means of its curvature in terms of direction, and the problem is completely solved.

7. In some problems it may be of interest to have the distance of the line of action of the resultant pressure from the mid-line of the barrier. This is found as follows. Using R_x, R_y, M as already defined in §2, the moment of the pressure about D is

$$M - x_D R_y + y_D R_x$$

in the anticlockwise sense in our figure. Hence by (8), (16) the distance of the resultant pressure measured from D towards A is

$$\frac{\frac{1}{8}a_4 + (a_3 - a_1)\sin\sigma_0 - 2a_2\sin^2\sigma_0 - \frac{1}{2}a_1^3\sin\sigma_0 - \frac{3}{8}a_1^2a_2 + (a_2 + 4a_1\sin\sigma_0)x_D - a_1^2 y_D}{\left\{a_1^4 + (a_2 + 4a_1\sin\sigma_0)^2\right\}^{1/2}} \quad \ldots \quad (23)$$

It may also be useful to know that the resultant pressure meets the line through D perpendicular to the direction of the stream at distance

$$\frac{1}{8}\frac{a_4}{a_1^2} + \left(\frac{a_3 - a_1}{a_1^2} - \frac{1}{2}a_1\right)\sin\sigma_0 - 2\frac{a_2}{a_1^2}\sin\sigma_0 - \frac{3}{8}a_2 + \frac{a_2 + 4a_1\sin\sigma_0}{a_1^2}x_D - y_D \quad \ldots \quad (24)$$

from D, measured positive on the side of A.

8. In the case of the problem defined by (4), so that the barrier consists of two planes meeting at C at an angle $\lambda\pi$, whose bisector makes an angle σ_0 with the direction of the stream at infinity, we have

$$a_1 = 2\lambda\cos\sigma_0, \quad a_2 = -2\lambda\sin 2\sigma_0, \quad a_3 = 2\lambda\cos 3\sigma_0, \quad a_4 = -2\lambda\sin 4\sigma_0, \ldots .$$

Equations (8), (16) give

$$R_x = \pi\lambda^2\cos^2\sigma_0, \quad R_y = \pi\lambda\sin\sigma_0\cos\sigma_0, \quad M = \frac{\pi\lambda}{4}\sin\sigma_0\cos\sigma_0\left\{2(1-\lambda^2)\cos^2\sigma_0 - 1\right\}.$$

Further, substituting for ζ in (18), (20) we find

$$x_D = \frac{1}{4} - \int_0^1 \left\{ \frac{1-\rho^2}{4\rho^3}\left[(1-2\rho\sin\sigma_0+\rho^2)\cos\left(2\lambda\tan^{-1}\frac{\rho\cos\sigma_0}{1-\rho\sin\sigma_0}\right) + (1+2\rho\sin\sigma_0+\rho^2)\cos\left(2\lambda\tan^{-1}\frac{\rho\cos\sigma_0}{1+\rho\sin\sigma_0}\right)\right] - \frac{1}{2\rho^3} + \frac{\lambda^2\cos^2\sigma_0}{\rho}\right\} d\rho ,$$

$$y_D = -\int_0^1 \left\{ \frac{1-\rho^2}{4\rho^3}\left[(1-2\rho\sin\sigma_0+\rho^2)\sin\left(2\lambda\tan^{-1}\frac{\rho\cos\sigma_0}{1-\rho\sin\sigma_0}\right) - (1+2\rho\sin\sigma_0+\rho^2)\sin\left(2\lambda\tan^{-1}\frac{\rho\cos\sigma_0}{1+\rho\sin\sigma_0}\right)\right] + \frac{\lambda\sin\sigma_0\cos\sigma_0}{\rho}\right\} d\rho ;$$

$$\tfrac{1}{2}(x_A - x_{A'}) = -\sin\sigma_0 - \tfrac{1}{2}\pi\lambda\sin\sigma_0\cos\sigma_0 - \int_0^1 \left\{ \frac{1-\rho^2}{4\rho^3}\left[(1-2\rho\sin\sigma_0+\rho^2)\cos\left(2\lambda\tan^{-1}\frac{\rho\cos\sigma_0}{1-\rho\sin\sigma_0}\right) - (1+2\rho\sin\sigma_0+\rho^2)\cos\left(2\lambda\tan^{-1}\frac{\rho\cos\sigma_0}{1+\rho\sin\sigma_0}\right) + \frac{\sin\sigma_0}{\rho^2}\right\} d\rho ,$$

$$\tfrac{1}{2}(y_A - y_{A'}) = \lambda\cos\sigma_0 + \tfrac{1}{2}\pi\lambda^2\cos^2\sigma_0 - \int_0^1 \left\{ \frac{1-\rho^2}{4\rho^3}\left[(1-2\rho\sin\sigma_0+\rho^2)\sin\left(2\lambda\tan^{-1}\frac{\rho\cos\sigma_0}{1-\rho\sin\sigma_0}\right) + (1+2\rho\sin\sigma_0+\rho^2)\sin\left(2\lambda\tan^{-1}\frac{\rho\cos\sigma_0}{1+\rho\sin\sigma_0}\right) - \frac{\lambda\cos\sigma_0}{\rho^2}\right\} d\rho$$

the integrands in the integrals of x_D, y_D being zero when $\rho = 0$, while those in $\frac{1}{2}(x_A - x_{A'}), \frac{1}{2}(y_A - y_{A'})$ have the values

$$\sin\sigma_0, \quad \tfrac{1}{6}(\lambda - \lambda^3)\cos 3\sigma_0 - \tfrac{1}{2}(\lambda + \lambda^3)\cos\sigma_0$$

when $\rho = 0$. The knowledge of these values of the integrands is useful when numerical or graphical integration has to be resorted to.

Starting with any point in the z plane as D, and with a given direction of the x axis, namely that of the stream at infinity, x_D, y_D fix the origin, $\frac{1}{2}(x_A - x_{A'})$, $\frac{1}{2}(y_A - y_{A'})$ fix the points A, A'. The angles that CA, CA' make with the positive direction of the x axis are

$$\sigma_0 + \tfrac{1}{2}\lambda\pi, \quad \sigma_0 - \tfrac{1}{2}\lambda\pi.$$

We can therefore draw CA, CA' and the geometry of the z plane is completely determined. The line of action of the resultant pressure makes an angle

$$\tan^{-1}\left(\tfrac{1}{\lambda}\tan\sigma_0\right)$$

with the x axis, and is at distance

$$\tfrac{1}{4}\sin\sigma_0\ \frac{2(1-\lambda^2)\cos^2\sigma_0-1}{(\sin^2\sigma_0+\lambda^2\cos^2\sigma_0)^{1/2}}$$

from the origin, in such a sense that the moment is anticlockwise when this distance is positive.

9. A curved barrier of some interest is that defined by

$$\zeta=\left(\frac{1+e^{i\sigma_0\tau}}{1-e^{-i\sigma_0\tau}}\right)\left(\frac{1+t_1e^{i\sigma_1\tau}}{1-t_1e^{-i\sigma_1\tau}}\right)^{-\frac{\cos\sigma_0}{\cos\sigma_1\sin\alpha_1}},$$

where $t_1,\alpha_1,\sigma_0,\sigma_1$ are connected by the relations

$$t_1=\tan\frac{\alpha_1}{2}\ ,\qquad \sin\sigma_0=\sin\alpha_1\sin\sigma_1 .$$

Since the value of λ is unity, the barrier is continuous at C. For different values of α_1,σ_1 we get first approximations to barriers in the form of elliptic cylinders of various eccentricities and in different orientations relatively to the direction of the stream at infinity. We now have

$$a_1=2\cos\sigma_0-\frac{2\cos\sigma_0}{\cos\sigma_1\sin\alpha_1}t_1\cos\sigma_1\ ,\quad a_2=-2\sin2\sigma_0+\frac{2\cos\sigma_0}{\cos\sigma_1\sin\alpha_1}t_1^2\sin2\sigma_1\ ,\ \ldots$$

and equations (8), (16) give

$$R_x=\pi\cos^2\sigma_0\left(\frac{\cos\alpha_1}{1+\cos\alpha_1}\right)^2,\quad R_y=\pi\sin\sigma_0\cos\sigma_0\left(\frac{\cos\alpha_1}{1+\cos\alpha_1}\right)^2,$$

$$M=-\tfrac{1}{4}\pi\sin\sigma_0\cos\sigma_0\ \frac{\cos^2\alpha_1}{(1+\cos\alpha_1)^4}\left(1+8\cos\alpha_1+\cos^2\alpha_1+4\sin^2\sigma_0\right);$$

while the line of action of the resultant pressure is at distance

$$-\tfrac{1}{4}\frac{\sin\sigma_0}{(1+\cos\alpha_1)^2}\left(1+8\cos\alpha_1+\cos^2\alpha_1+4\sin^2\sigma_0\right)$$

from the origin, in such a sense that the moment is anticlockwise when this distance is positive.

The geometry of the z plane is given by the direction θ at any point of the barrier, and the corresponding radius of curvature $\frac{R}{k}$ both expressed in terms of σ as a parameter:-

$$\theta=\pm\frac{\pi}{2}+\sigma_0-\frac{\cos\sigma_0}{\cos\sigma_1\sin\alpha_1}\left\{\tan^{-1}\left(\frac{t_1\sin(\sigma+\sigma_1)}{1+t_1\cos(\sigma+\sigma_1)}\right)+\tan^{-1}\left(\frac{t_1\sin(\sigma-\sigma_1)}{1-t_1\cos(\sigma-\sigma_1)}\right)\right\},$$

$\pm$ according as $\sigma\gtrless\sigma_0$; and

$$R=-\frac{2(1+\cos\overline{\sigma+\sigma_0})}{\cos\sigma_0\cos\alpha_1}\left(1+\sin\alpha_1\cos\overline{\sigma+\sigma_1}\right)\left(1-\sin\alpha_1\cos\overline{\sigma-\sigma_1}\right)\left(\frac{1+\sin\alpha_1\cos\overline{\sigma+\sigma_1}}{1-\sin\alpha_1\cos\overline{\sigma-\sigma_1}}\right)^{-\frac{\cos\sigma_0}{2\cos\sigma_1\sin\alpha_1}}.$$

AUTOGRAPHE

de

Brodetzky

Selig

OBSERVATIONS

M.s. /zum Teile eigenhändig /, wissenschaftliche Arbeit, mit dem Titel „The Line of Aktion of the Resultant Pressure in Discontinous Fluid Motion" /Die Kraftlinien des resultierenden Druckes bei diskontinuierlicher Flüssigkeitsbewegung/. Communicated by Sir George Greenhill." Abgedruckt in den „Proceedings am 18.10. 1922.

B R O D E T S K Y , Selig, Dr. Phil., M a t h e m a t i k e r .

Geb. 10.Feber 1888,Olviopol, Ukraine.

Stud. 1906-1911 Cambridge, 1912-13 Leipzig, Dr.Phil. 1913. 1914 - 1919 Lekt., angew.Math.,Univ. Bristol, 19 - 24 Reader, seit 1924 Prof. d. angew. Math. Univ. Leeds,Engl.

Schrieb: First course in nomography,135 p.,London 1920. Mechanical principles of the aeroplane, 272 p., London 1921.

Aeronaut.Journ.: Long tails versus short tails in aerodynamics, 6 p. Longitud. initial mot. & forced oscillats of a disturbed aeroplane, m.G.H.Bryan;18 p. Astron.Nachr. The problem of the resisting medium, Dictionary of applic.Phys. Nomography, 8 p.

AUTOGRAPHE

de

Brodetzky

Leeds

OBSERVATIONS

M.s. /z
senscha
,, The
tant Pr
Motion
tierend
licher
Communi
Abgedru
am 18.1

Edinburgh, Math. Soc. Proc. Linear differential equat. of the 2. ordre, 16 p., London Roy. Soc. Proc. The two-dimensional mot. of a plane lamina in a resisting medium, 17 p. The line of act. of the resultant pressure in discontinuous fluid mot, 12 p. Math. Congress 1912: Integrals in dynamics & the problem of 3 bodies, 15 p., Math. Gaz. Momography, 12 p. Elementary theory of statical stability, 4 p. etc. Messenger of. Math. Some theorems concerning spherical harmonics, 24 p. Nature: Gilding or wind flight, 2 p. Paris, Acad. C. R.: Une analogie entre l. equat. linéaires différentielles & l. équat. algébr., 3 p. Phil. Mag., Quart. Journ. of pure a. appl. Math., Science progress, Tohoku Math. Journ. u. A. m.

DSI